

Algorithmic aspects of disjunctive domination in graphs

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Abstract

For a graph $G = (V, E)$, a set $D \subseteq V$ is called a *disjunctive dominating set* of G if for every vertex $v \in V \setminus D$, v is either adjacent to a vertex of D or has at least two vertices in D at distance 2 from it. The cardinality of a minimum disjunctive dominating set of G is called the *disjunctive domination number* of graph G , and is denoted by $\gamma_2^d(G)$. The MINIMUM DISJUNCTIVE DOMINATION PROBLEM (MDDP) is to find a disjunctive dominating set of cardinality $\gamma_2^d(G)$. Given a positive integer k and a graph G , the DISJUNCTIVE DOMINATION DECISION PROBLEM (DDDP) is to decide whether G has a disjunctive dominating set of cardinality at most k . In this article, we first propose a linear time algorithm for MDDP in proper interval graphs. Next we tighten the NP-completeness of DDDP by showing that it remains NP-complete even in chordal graphs. We also propose a $(\ln(\Delta^2 + \Delta + 2) + 1)$ -approximation algorithm for MDDP in general graphs and prove that MDDP can not be approximated within $(1 - \epsilon) \ln(|V|)$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME}(|V|^{O(\log \log |V|)})$. Finally, we show that MDDP is APX-complete for bipartite graphs with maximum degree 3.

Keywords: Domination, Chordal graph, Graph algorithm, Approximation algorithm, NP-complete, APX-complete.

1 Introduction

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $N_G(v) = \{u \in V | uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the open neighborhood and the closed neighborhood of v , respectively. For two distinct vertices $u, v \in V$, the distance $\text{dist}_G(u, v)$ between u and v is the length of a shortest path between u and v . A vertex u dominates v if either $u = v$ or u is adjacent to v . A set $D \subseteq V$ is called a *dominating set* of $G = (V, E)$ if each $v \in V$ is dominated by a vertex in D , that is, $|N_G[v] \cap D| \geq 1$ for all $v \in V$. The domination number of a graph G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . For a graph G , the MINIMUM DOMINATION problem is to find a dominating set of cardinality $\gamma(G)$. Domination in graphs is one of the classical problems in graph theory and it has

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been well studied from theoretical as well as algorithmic point of view [9, 10]. Over the years, many variants of domination problem have been studied in the literature due to its application in different fields varying from computer science to electrical engineering, operation research to network securities etc. The concept of *disjunctive domination* is a recent and an interesting variation of domination [8].

In domination problem, our goal is to place minimum number of sentinels at some vertices of the graph so that all the remaining vertices are adjacent to at least one sentinel. In practice, depending upon the monitoring power, we can have different types of sentinels. To secure the graph with different types of sentinels, we need concept of different variants of domination. Efforts made in this direction have given rise to different types of domination, such as, distance domination, exponential domination, secondary domination. In some cases, it might happen that the monitoring power of a sentinel is inversely proportional to the distance, that is, the domination power of a vertex reduces as the distance increases. Motivated by this idea, Goddard et al. [8] have introduced the concept of *disjunctive domination* which captures the notion of decay in domination with increasing distance. A set $D_d \subseteq V$ is called a *b-disjunctive dominating set* of G if every vertex $v \in V \setminus D_d$ is either adjacent to a vertex in D_d or there are at least b vertices of D_d within a distance of two from v . The minimum cardinality of a *b-disjunctive dominating set* of G is called the *b-disjunctive domination number* and it is denoted by $\gamma_b^d(G)$. A vertex v is said to be *b-disjunctively dominated* by $D_d \subseteq V$ if either $v \in D_d$ or v is adjacent to a vertex of D_d or has at least b vertices in D_d at distance 2 from it. Note that disjunctive domination is more general concept than distance two domination, since the parameter $\gamma_1^d(G)$ is the distance two domination number. For simplicity, 2-disjunctive domination is called disjunctive domination. The disjunctive domination problem and its decision version are defined as follows:

MINIMUM DISJUNCTIVE DOMINATION PROBLEM (MDDP)

Instance: A graph $G = (V, E)$.

Solution: A disjunctive dominating set D_d of G .

Measure: Cardinality of the set D_d .

DISJUNCTIVE DOMINATION DECISION PROBLEM (DDDP)

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Does there exist a disjunctive dominating set D_d of G such that $|D_d| \leq k$?

The concept of disjunctive domination has been introduced recently in 2014 [8] and further studied in [11]. In [8], Goddard et al. have proven bounds on disjunctive domination number for specially regular graphs and claw-free graphs. They have shown that finding minimum *b-disjunctive dominating set* problem is NP-complete for planar and bipartite graphs and also designed a dynamic programming based linear time algorithm to find a minimum *b-disjunctive dominating set* in a tree. In [11], Henning et al. have studied the relation between domination number and disjunctive domination number of a tree T and proved that $\gamma(T) \leq 2\gamma_2^d(T) - 1$. They have also given a constructive characterization of the trees achieving equality in this bound. On the other hand, a variation of disjunctive domination is also studied in the literature (see [12]).

In this paper, our focus is on algorithmic study of disjunctive domination problem. The rest of the paper is organized as follows. In Section 2, we give some pertinent definitions and notations that would be used in the rest of the paper. In this section, we also observe some graph classes where domination problem is NP-complete but disjunctive domination can be easily solved and vice versa. This motivates us to study the status of the problem in other graph classes. In Section 3, we design a linear time algorithm

for disjunctive domination problem in proper interval graphs, an important subclass of chordal graphs. In Section 4, we prove that DDDP remains NP-complete for chordal graphs. In Section 5, we design a polynomial time approximation algorithm for MDDP for general graph G with approximation ratio $\ln(\Delta^2 + \Delta + 2) + 1$, where Δ is the maximum degree of G . In this section, we also prove that MDDP can not be approximated within $(1 - \epsilon) \ln(|V|)$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME}(|V|^{O(\log \log |V|)})$. In addition, for bipartite graphs with maximum degree 3, MDDP is shown to be APX-complete in this section. Finally, Section 6 concludes the paper.

2 Preliminaries

2.1 Notations

Let $G = (V, E)$ be a graph. Let $N_G^2(v)$ denote the set of vertices which are at distance 2 from the vertex v in graph G . Let $G[S]$, $S \subseteq V$ denote the induced subgraph of G on the vertex set S . The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of neighbors of v , that is, $d_G(v) = |N_G(v)|$. The *minimum degree* and *maximum degree* of a graph G is defined by $\delta(G) = \min_{v \in V(G)} d_G(v)$ and $\Delta(G) = \max_{v \in V(G)} d_G(v)$, respectively. A set $S \subseteq V$ is called an independent set of a graph $G = (V, E)$ if $uv \notin E$ for all $u, v \in S$. A set $K \subseteq V$ is called a clique of a graph $G = (V, E)$ if $uv \in E$ for all $u, v \in K$. A set $C \subseteq V$ is called a *vertex cover* of a graph $G = (V, E)$ if for each edge $ab \in E$, either $a \in C$ or $b \in C$. Let n and m denote the number of vertices and number of edges of G , respectively. In this paper, we only consider connected graphs with at least two vertices.

2.2 Graph Classes

A graph G is said to be a *chordal graph* if every cycle in G of length at least four has a chord, that is, an edge joining two non-consecutive vertices of the cycle. Let \mathcal{F} be a family of sets. The intersection graph of \mathcal{F} is obtained by taking each set in \mathcal{F} as a vertex and joining two sets in \mathcal{F} if and only if they have a non-empty intersection. A graph G is an *interval graph* if G is the intersection graph of a family \mathcal{F} of intervals on the real line. A graph G is called a *proper interval graph* if it is the intersection graph of a family \mathcal{F} of intervals on the real line such that no interval in \mathcal{F} contains another interval in \mathcal{F} set theoretically. A vertex $v \in V(G)$ is a *simplicial vertex* of G if $N_G[v]$ is a clique of G . An ordering $\alpha = (v_1, v_2, \dots, v_n)$ is a *perfect elimination ordering* (PEO) of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \dots, v_n\}]$ for all i , $1 \leq i \leq n$. A graph G has a PEO if and only if G is chordal [7]. A PEO $\alpha = (v_1, v_2, \dots, v_n)$ of a chordal graph is a *bi-compatible elimination ordering* (BCO) if $\alpha^{-1} = (v_n, v_{n-1}, \dots, v_1)$, i.e., the reverse of α , is also a PEO of G . A graph G has a BCO if and only if G is a proper interval graph [14]. A graph $G = (V, E)$ is called a *split graph* if its vertex set, V , can be partitioned into two sets, say X and Y , such that X is an independent set and Y is a clique of G .

2.3 Domination vs disjunctive domination

In this subsection, we make some observations on complexity difference of domination and disjunctive domination problem. It is known that domination problem is NP-complete for split graphs [4] and for graphs with diameter two [2]. But disjunctive domination problem can be easily solved in these graph classes. Because, disjunctive domination number is at most 2 in these classes and $\gamma_2^d(G) = 1$ if and only if G contains a vertex of degree $n - 1$. Next, we define a graph class, called *GC graph*, for which domination problem is easily solvable, but disjunctive domination problem is NP-complete.

Definition 2.1 (GC graph). A graph $G' = (V', E')$ is said to be a GC graph if it can be constructed from a general graph $G = (V, E)$ by adding a pendant vertex to every vertex of G . Formally, $V' = V \cup \{w_i \mid 1 \leq i \leq n\}$ and $E' = E \cup \{v_i w_i \mid 1 \leq i \leq n\}$.

Note that, every vertex of a GC graph G' is either a pendant vertex or adjacent to a unique pendant vertex and hence, $\gamma(G') = n$. In Section 4, we show that DDDP is NP-complete for the class of GC graphs.

3 Polynomial time algorithm for proper interval graphs

In this section, we present a polynomial time algorithm to find a minimum cardinality disjunctive dominating set in proper interval graphs.

Let $\alpha = (v_1, v_2, \dots, v_n)$ be a BCO of the proper interval graph G . Let $MaxN_G(v_i)$ denote the maximum index neighbor of v_i with respect to the ordering α . We start with an empty set D . At each iteration i of the algorithm, we update the set D in such a way that the vertex v_i and all the vertices which appear before v_i in the BCO α , are disjunctively dominated by the set D . At the end of n^{th} iteration, D disjunctively dominate all the vertices of graph G . The algorithm DISJUNCTIVE-PIG for finding a minimum cardinality disjunctive dominating set in a proper interval graph is given below.

Algorithm 1: DISJUNCTIVE-PIG($G, \alpha = (v_1, v_2, \dots, v_n)$)

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Initialize  $D = \emptyset$ ;
for  $i = 1 : n$  do
    Compute  $N_G(v_i) \cap D$  and  $N_G^2(v_i) \cap D$ ;
    Case 1: Either  $N_G[v_i] \cap D \neq \emptyset$ , or  $|N_G^2(v_i) \cap D| \geq 2$ 
        No update in  $D$  is done;
    Case 2:  $N_G[v_i] \cap D = \emptyset$  and  $N_G^2(v_i) \cap D = \emptyset$ 
        Update  $D$  as  $D = D \cup \{MaxN_G(v_i)\}$ ;
    Case 3:  $N_G[v_i] \cap D = \emptyset$  and  $|N_G^2(v_i) \cap D| = 1$ 
        Find  $v_r \in N_G^2(v_i) \cap D$ ;
         $v_j = Max[v_i]$ ;  $v_k = Max[v_j]$ ;
         $S = \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ ;
        Subcase 3.1: For every  $v \in S$ , either  $vv_k \in E$  or  $d(v, v_r) = 2$ 
            Update  $D$  as  $D = D \cup \{v_k\}$ ;
        Subcase 3.2:  $v_s$  is the least index vertex in  $S$  such that
             $d(v_s, v_k) = 2$  and  $d(v_s, v_r) > 2$ 
            Update  $D$  as  $D = D \cup \{MaxN_G(v_s)\}$ ;
return  $D$ ;

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Next we give the proof of correctness of the algorithm. Let $\alpha = (v_1, v_2, \dots, v_n)$ be the BCO of a proper interval graph G . Define the set $V_i = \{v_1, v_2, \dots, v_i\}$, $1 \leq i \leq n$, and $V_0 = \emptyset$. Also suppose that D_i denotes the set D obtained after processing vertex v_i , $1 \leq i \leq n$, and $D_0 = \emptyset$. We will prove that D_n is a minimum cardinality disjunctive dominating set of G .

Theorem 3.1. For each i , $0 \leq i \leq n$, the following statements are true:

- (a) D_i disjunctively dominates the set V_i .
- (b) There exists a minimum cardinality disjunctive dominating set D_d^* such that D_i is contained in D_d^* .

Proof. We prove the theorem by induction on i . The basis step is trivial as $D_0 = \emptyset$. Next assume that the theorem is true for $i - 1$. So, (a) D_{i-1} disjunctively dominates the set V_{i-1} , (b) there exists a minimum cardinality disjunctive dominating set D_d^* such that D_{i-1} is contained in D_d^* .

Next we prove the theorem for i . According to our algorithm, we need to discuss the following three cases.

Case 1: Either $N_G[v_i] \cap D_{i-1} \neq \emptyset$, or $|N_G^2(v_i) \cap D_{i-1}| \geq 2$.

Here $D_i = D_{i-1}$. It is easy to notice that all the conditions of the theorem are satisfied.

Case 2: $N_G[v_i] \cap D_{i-1} = \emptyset$ and $N_G^2(v_i) \cap D_{i-1} = \emptyset$.

Here $D_i = D_{i-1} \cup \{v_j\}$ where $v_j = \text{Max}N_G(v_i)$. Hence, condition (a) of the theorem is trivially satisfied. If $v_j \in D_d^*$, then $D_i \subseteq D_d^*$. Hence both the conditions of the theorem are satisfied, and D_d^* is the required minimum cardinality disjunctive dominating set of G . If $v_j \notin D_d^*$, then there are two possibilities:

(I) There exists a vertex $v_p \in N_G[v_i] \cap D_d^*$.

Define the set $D_d^{**} = (D_d^* \setminus \{v_p\}) \cup \{v_j\}$. Note that $D_i \subseteq D_d^{**}$, and $|D_d^*| = |D_d^{**}|$. Now, to prove condition (b) of the theorem, it is enough to show that D_d^{**} is a disjunctive dominating set of G . Note that $D_{i-1} \cup \{v_j\} \subseteq D_d^{**}$. Now consider an arbitrary vertex v_a of G . If $a < i$, then the vertex v_a is disjunctively dominated by the set D_{i-1} , and hence by D_d^{**} . If $a \geq i$, and $v_p \in N_G[v_a]$, then $v_j \in N_G[v_a]$. If $a \geq i$, and $v_p \in N_G^2(v_a)$, then $v_j \in N_G[v_a]$ or $v_j \in N_G^2(v_a)$. This proves that D_d^{**} is a disjunctive dominating set of G .

(II) For $q < s$, vertices $v_q, v_s \in N_G^2(v_i) \cap D_d^*$.

Let $\text{Max}N_G(v_i) = v_j$ and $\text{Max}N_G(v_j) = v_k$. Then $q < s \leq k$. Let $v_t = \text{Max}N_G(v_s)$ and $v_r = \text{Max}N_G(v_t)$. We again consider three possibilities:

(i) $q < s < i$

Here $r \leq j$. Now consider an arbitrary vertex v_a of G . If $a < i$, then the vertex v_a is disjunctively dominated by the set D_{i-1} . If $a \geq i$, and $v_s \in N_G^2(v_a)$ or $v_q, v_s \in N_G^2(v_i)$, then $v_j \in N_G[v_a]$. Hence $(D_d^* \setminus \{v_q, v_s\}) \cup \{v_j\}$ is a disjunctive dominating set of G of cardinality less than $|D_d^*|$, which is a contradiction, as D_d^* is a minimum disjunctive dominating set of G . Therefore, this situation will never arise.

(ii) $q < i < s$

Consider an arbitrary vertex v_a of G . If $a < i$, then the vertex v_a is disjunctively dominated by the set D_{i-1} . If $a \geq i$, and $v_q \in N_G^2(v_a)$, then $v_j \in N_G[v_a]$. If $a \geq i$, and $v_q \notin N_G^2(v_a)$, and either $v_s \in N_G[v_a]$ or $v_s \in N_G^2(v_a)$, then either $v_j \in N_G[v_a]$ or $v_t \in N_G[v_a]$. Hence, if we define $D_d^{**} = (D_d^* \setminus \{v_q, v_s\}) \cup \{v_j, v_t\}$, then D_d^{**} is a minimum cardinality disjunctive dominating set of G and $D_i \subseteq D_d^{**}$. This proves the condition (b) of the theorem.

(iii) $i < q < s$

Here $s \leq k$. Consider an arbitrary vertex v_a of G . If $a < i$, then the vertex v_a is disjunctively dominated by the set D_{i-1} . If $a \geq i$, and $v_q \in N_G[v_a]$ or $v_s \in N_G[v_a]$ or $v_q, v_s \in N_G^2(v_a)$ or $v_s \in N_G^2(v_a)$, then either $v_j \in N_G[v_a]$ or $v_t \in N_G[v_a]$. Hence, if we define $D_d^{**} = (D_d^* \setminus \{v_q, v_s\}) \cup \{v_j, v_t\}$, then D_d^{**} is a minimum cardinality disjunctive dominating set of G and $D_i \subseteq D_d^{**}$. This proves the condition (b) of the theorem.

Case 3: $|N_G^2(v_i) \cap D_{i-1}| = 1$, $N_G^2(v_i) \cap D_{i-1} = \{v_r\}$ ($r < i$), $v_j = \text{Max}N_G(v_i)$, $v_k = \text{Max}N_G(v_j)$, and $S = \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$.

Subcase 3.1: For every $v \in S$, either $vv_k \in E$ or $d(v, v_r) = 2$.

Here $D_i = D_{i-1} \cup \{v_k\}$.

Clearly, condition (a) of the theorem is satisfied. If $v_k \in D_d^*$, then $D_i \subseteq D_d^*$. Hence both the conditions of the theorem are satisfied, and D_d^* is the required minimum cardinality disjunctive dominating set of G . If $v_k \notin D_d^*$, then to disjunctively dominate v_i , at least one vertex before v_k in BCO α , say v_p , must belong to D_d^* . Define $D_d^{**} = (D_d^* \setminus \{v_p\}) \cup \{v_k\}$. Then $|D_d^{**}| = |D_d^*|$ and $D_i \subseteq D_d^{**}$. Now, to

prove condition (b) of the theorem, it is enough to show that D_d^{**} is a disjunctive dominating set of G . Consider an arbitrary vertex v_b in G . If $b \leq k$, then v_b is disjunctively dominated by the set $D_{i-1} \cup \{v_k\}$, and hence by D_d^{**} . If $b > k$, and $v_p \in N_G[v_b]$, then $v_k \in N_G[v_b]$. If $b > k$, and $v_p \in N_G^2(v_b)$, then either $v_k \in N_G[v_b]$ or $v_k \in N_G^2(v_b)$. Hence D_d^{**} is a disjunctive dominating set of G .

Subcase 3.2: v_s is the least index vertex in S such that $d_G(v_s, v_k) = 2$ and $d(v_s, v_r) > 2$.

Here $D_i = D_{i-1} \cup \{v_p\}$, where $v_p = \text{Max}N_G(v_s)$. Clearly, condition (a) of the theorem is trivially satisfied. If $v_p \in D_d^*$, then $D_i \subseteq D_d^*$. Hence both the conditions of the theorem are satisfied, and D_d^* is the required minimum cardinality disjunctive dominating set of G . If $v_p \notin D_d^*$, then there are two possibilities:

(I) $v_q \in D_d^* \setminus D_{i-1}$, where $q < p$

Define $D_d^{**} = (D_d^* \setminus \{v_q\}) \cup \{v_p\}$. Then $|D_d^{**}| = |D_d^*|$ and $D_i \subseteq D_d^{**}$. Now, to prove condition (b) of the theorem, it is enough to show that D_d^{**} is a disjunctive dominating set of G . Consider an arbitrary vertex v_b in G . If $b < i$, then v_b is disjunctively dominated by the set D_{i-1} , and hence by D_d^{**} . If $b \geq i$, and $v_q \in N_G[v_b]$, then $v_p \in N_G[v_b]$ or $v_r, v_p \in N_G^2(v_b)$. If $b \geq i$, and $v_q \in N_G^2(v_b)$, then $v_p \in N_G[v_b]$ or $v_p \in N_G^2(v_b)$. Hence D_d^{**} is a disjunctive dominating set of G .

(II) $D_d^* \cap V_p = D_{i-1}$

To disjunctively dominate the vertex v_s , at least two vertices from the set $V_w \setminus (V_p \cup D_{i-1})$ must belong to D_d^* , where $v_w = \text{Max}N_G(v_p) = \text{Max}N_G(\text{Max}N_G(v_s))$. Let them be v_{t1}, v_{t2} where $t_1 < t_2$. Note that $p < t_1 < t_2 \leq w$. Let $v_{w'} = \text{Max}N_G(\text{Max}N_G(v_w))$. Now define the set $D_d^{**} = (D_d^* \setminus \{v_{t1}, v_{t2}\}) \cup \{v_p, v_{w'}\}$. Then $|D_d^{**}| = |D_d^*|$ and $D_i \subseteq D_d^{**}$. Now to prove the condition (b) of the theorem, it is enough to show that D_d^{**} is a disjunctive dominating set of G . Consider an arbitrary vertex v_b in G . If $b < i$, then v_b is disjunctively dominated by the set D_{i-1} , and hence by D_d^{**} . If $s > b \geq i$, then either $v_p \in N_G[v_b]$ or $v_r, v_p \in N_G^2(v_b)$ (since every vertex in $V_{s-1} \setminus V_{i-1}$ is at distance 2 from the vertex v_r). If $b \geq s$, and $v_{t1} \in N_G[v_b]$ or $v_{t2} \in N_G[v_b]$ or $v_{t1}, v_{t2} \in N_G^2(v_b)$, then either $v_p \in N_G[v_b]$ or $v_p, v_{w'} \in N_G^2(v_b)$. If $b \geq s$ and $v_{t1} \in N_G^2(v_b)$ and $v_{t2} \notin N_G^2(v_b)$, then $v_p \in N_G[v_b]$. If $b \geq s$ and $v_{t1} \notin N_G^2(v_b)$ and $v_{t2} \in N_G^2(v_b)$, then either $v_{w'} \in N_G[v_b]$ or $v_p, v_{w'} \in N_G^2(v_b)$. Hence D_d^{**} is a disjunctive dominating set of G . Hence our theorem is proved. \square

In view of the above theorem, the set D computed by the algorithm DISJUNCTIVE-PIG is a minimum cardinality disjunctive dominating set of G . Now, we show that the algorithm DISJUNCTIVE-PIG can be implemented in polynomial time. We use the adjacency list representation of the graph. We maintain an array D_{set} for the set D such that $D_{set}[j] = 1$ if $v_j \in D$. We maintain the All pair distance Matrix $Dist[1..n, 1..n]$ such that $Dist[i, j]$ is the distance between v_i and v_j . This can be done in $O(n^3)$ time. Now $N_G[v_i] \cap D$ can be computed in $O(n)$ time by looking up $Dist$ matrix and array D_{set} . Similarly, $N_G^2(v_i) \cap D$ can be computed in $O(n)$ time. Also $\text{Max}N_G(v_i)$ can be computed in $O(n)$ time. Hence, in any iteration, all the operations can be done in $O(n^2)$ time. Therefore overall time is $O(n^3)$, as number of iterations are n . Since, BCO of a proper interval graph can be computed in $O(n + m)$ time [15], and all the computations in the algorithm DISJUNCTIVE-PIG can be done in $O(n^3)$ time, we have the following theorem.

Theorem 3.2. *MDDP can be solved in $O(n^3)$ time in proper interval graphs.*

However, the algorithm DISJUNCTIVE-PIG can be implemented in $O(n + m)$ time using additional data structures. The details are given below. We first describe some notations. Let $\alpha = (v_1, v_2, \dots, v_n)$ be a BCO of the proper interval graph $G = (V, E)$. We maintain a set D . Initially $D = \emptyset$. At the end of n^{th} iteration, D becomes a minimum cardinality disjunctive dominating set of G . We maintain two arrays $Min[1, \dots, n]$ and $Max[1, \dots, n]$. For a vertex v , $Min[v]$ denotes the minimum index vertex in the BCO α , which is adjacent to v , and $Max[v]$ denotes the maximum index vertex in the BCO α , which

is adjacent to v . We also maintain an array $D_{count}[1, \dots, n]$. For a vertex $v \in V$, $D_{count}[v]$ denotes the number of vertices in D which dominate the vertex v .

Lemma 3.3. *The following statements are true:*

- (i) $D_{count}[v_i] = |N_G[v_i] \cap D|$.
- (ii) If $N_G[v_i] \cap D = \emptyset$, then $D_{count}[Max[v_i]] + D_{count}[Min[v_i]] = |N_G^2(v_i) \cap D|$.

Proof. The proof is easy and hence is omitted. □

Based on the above discussion, we have the detailed algorithm for finding minimum cardinality disjunctive dominating set which is presented in M-DISJUNCTIVE-PIG.

Algorithm 2: M-DISJUNCTIVE-PIG(G)

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Obtain a BCO  $\sigma = \{v_1, v_2, \dots, v_n\}$  of proper interval graph  $G$ ;
Obtain array  $Min$  and  $Max$ ;
Initialize  $D = \emptyset$ ;
Initialize  $D_{count}[v_i] = 0$  for all  $i$ ,  $1 \leq i \leq n$ ;
for  $i = 1 : n$  do
    if  $((D_{count}[v_i] \neq 0) \text{ or } (D_{count}[Max[v_i]] + D_{count}[Min[v_i]] \geq 2))$  then
         $\perp$  no update;
    else if  $(D_{count}[v_i] == 0) \text{ and } (D_{count}[Max[v_i]] + D_{count}[Min[v_i]] == 0)$  then
         $v_k = Max[v_i]$ ;
         $D = D \cup \{v_k\}$ ;
        foreach  $v \in N_G[v_k]$  do
             $\perp D_{count}[v] = D_{count}[v] + 1$ ;
    else if  $((D_{count}[v_i] == 0) \text{ and } (D_{count}[Max[v_i]] + D_{count}[Min[v_i]] == 1))$  then
        (This basically means that  $D_{count}[Min[v_i]] = 1$ )
        Let  $v_t = Min[v_i]$ ,  $v_j = Max[v_i]$ , and  $v_k = Max[v_j]$ ;
        Let  $\{v_r\} = N_G[v_t] \cap D$ ;
        for  $s = i + 1 : j - 1$  do
            Let  $v_a = Min[v_k]$ ,  $v_b = Max[v_r]$ ,  $v_c = Min[v_s]$ ;
            if  $s < a$  and  $b < c$  then
                 $D = D \cup \{Max[v_s]\}$ ;
                foreach  $v \in N_G[Max[v_s]]$  do
                     $\perp D_{count}[v] = D_{count}[v] + 1$ ;
                break;
            if  $s == j$  then
                 $D = D \cup \{v_k\}$ ;
                foreach  $v \in N_G[v_k]$  do
                     $\perp D_{count}[v] = D_{count}[v] + 1$ ;
         $\perp$ 
     $\perp$ 
return  $D$ ;

```

Next we show that this algorithm M-DISJUNCTIVE-PIG can be implemented in $O(n+m)$ time. We first compute $Max[v_i]$ and $Min[v_i]$ for each v_i , $1 \leq i \leq n$. This takes $O(d_G(v_i))$ time for each vertex v_i . Hence arrays Min and Max can be computed in $O(n+m)$ time. We can find a vertex in $N_G[v_t] \cap D$ in $O(1)$ time by maintaining an array $B[1, \dots, n]$ of linked lists such that $B[i]$ contains all the vertices of $N_G[v_i] \cap D$. This is done by inserting v_j in the linked lists of v_j and all the neighbors of v_j whenever v_j is included in D . So maintaining this information takes $\sum_{v \in D} (d(v)) = O(n+m)$ time. Therefore, all the

computations in the algorithm M-DISJUNCTIVE-PIG can be done in $\sum_{i=1}^n (d_G(v_i)) + \sum_{v \in D} (d_G(v)) = O(n + m)$ time.

In view of this, we have the following theorem.

Theorem 3.4. *The algorithm M-DISJUNCTIVE-PIG can be implemented in $O(n + m)$ time and hence MDDP can be solved in $O(n + m)$ time in proper interval graphs.*

4 NP-completeness

In this section, we prove that DDDP is NP-complete for chordal graphs. For that, we first show that DDDP is NP-complete for GC graphs. To prove this NP-completeness result, we use a reduction from another variant of domination problem, namely *2-domination problem*. For a graph $G = (V, E)$, a set $D_2 \subseteq V$ is called *2-dominating set* if every vertex $v \in V \setminus D_2$ has at least two neighbors in D_2 . Given a positive integer k and a graph $G = (V, E)$, the 2-DOMINATION DECISION PROBLEM (2DDP) is to decide whether G has a 2-dominating set of cardinality at most k . It is known that 2DDP is NP-complete for chordal graphs [13]. The following lemma shows that DDDP is NP-complete for GC graphs.

Lemma 4.1. *DDDP is NP-complete for GC graphs.*

Proof. Clearly, DDDP is in NP for GC graphs. To prove the NP-hardness, we give a polynomial transformation from 2DDP for general graphs. Let $G = (V, E)$ and k be an instance of 2DDP. Given a graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$, we construct the graph $G' = (V', E')$ in the following way: $V' = V \cup \{w_i \mid 1 \leq i \leq n\}$ and $E' = E \cup \{v_i w_i \mid 1 \leq i \leq n\}$. Clearly G' is a GC graph and it can be constructed from G in polynomial time.

The following claim is enough to complete the proof of the theorem.

Claim 4.2. *G has a 2-dominating set of cardinality at most k if and only if G' has a disjunctive dominating set of cardinality at most k .*

Proof. (Proof of the claim) Let D_2 be a 2-dominating set of G of cardinality at most k . Clearly D_2 is a disjunctive dominating set of G' . Because every $v_i \in V'$ either is in D_2 or dominated by at least two vertices of D_2 and every $w_i \in V'$ is either dominated by $v_i \in D_2$ or contains at least two vertices from D_2 at a distance of two. Hence, G' has a disjunctive dominating set of cardinality at most k .

Conversely, suppose that D_d is a disjunctive dominating set of G' of cardinality at most k . Note that, every vertex of G' is either a pendant vertex or a support vertex. Also, the vertex set of graph G is exactly the set of all support vertices of G' . Let P be the set of pendant vertices of graph G' , i.e., $P = \{w_i \mid 1 \leq i \leq n\}$. If a pendant vertex $w_i \in D_d$, then the set $D'_d = (D_d \setminus \{w_i\}) \cup \{v_i\}$ still remains a disjunctive dominating set of G' of cardinality at most k . So, without loss of generality we assume that $D_d \cap P = \emptyset$. Now for every vertex $v_i \in V$, either $v_i \in D_d$ or $|N_G(v_i) \cap D_d| \geq 2$. If not, let there is a vertex $v_i \in V \setminus D_d$ such that $|N_G(v_i) \cap D_d| \leq 1$. This implies that the vertex $w_i \in V'$ is neither dominated nor has at least two vertices from D_d at a distance of two, contradicting the fact that D_d is a disjunctive dominating set of G' . Hence, D_d is a 2-dominating set of G of cardinality at most k . \square

Hence, it is proved that DDDP is NP-complete for GC graphs. \square

It is easy to observe that, if the graph G is chordal, then the constructed graph G' in Lemma 4.1 is also chordal. Hence, we have the main result of this section as a corollary.

Theorem 4.3. *DDDP is NP-complete for chordal graphs.*

5 Approximation results

5.1 Approximation algorithm

In this subsection, we propose a $(\ln(\Delta^2 + \Delta + 2) + 1)$ -approximation algorithm for MDDP. Our algorithm is based on the reduction from MDDP to the CONSTRAINED MULTISSET MULTICOVER (CMSMC) problem. We first recall the definition of the CONSTRAINED MULTISSET MULTICOVER problem.

Let X be a set and \mathcal{F} be a collection of subsets of X . The SET COVER problem is to find a smallest sub-collection, say \mathcal{C} of \mathcal{F} , such that \mathcal{C} covers all the elements of X , that is, $\cup_{S \in \mathcal{C}} S = X$. The CONSTRAINED MULTISSET MULTICOVER problem is a generalization of the SET COVER problem. In this problem, \mathcal{F} is the collection of multisets of X , that is, each element $x \in X$ occurs in a multiset $S \in \mathcal{F}$ with arbitrary multiplicity, and each element $x \in X$ has an integer coverage requirement r_x which specifies how many times x has to be covered. Note that each set $S \in \mathcal{F}$ is chosen at most once. So, for a given set X , a collection \mathcal{F} of multisets of X , and integer requirement r_x for each $x \in X$, the CMSMC problem is to find a smallest collection $\mathcal{C} \subseteq \mathcal{F}$, such that \mathcal{C} covers each element x in X at least r_x times. In the case, when r_x is constant for each $x \in X$, then \mathcal{C} is called a r_x -cover of X , and the CMSMC problem is to find a minimum cardinality r_x -cover of X .

Theorem 5.1. *The MINIMUM DISJUNCTIVE DOMINATION PROBLEM for a graph $G = (V, E)$ with maximum degree Δ can be approximated with an approximation ratio of $\ln(\Delta^2 + \Delta + 2) + 1$.*

Proof. Let us show the transformation from MDDP to the CMSMC problem.

Construction : Let $G = (V, E)$ be a graph with n vertices and m edges where $V = \{v_1, v_2, \dots, v_n\}$ (an instance of MDDP). Now we construct an instance of the CMSMC problem, that is, a set X , a family \mathcal{F} of multisets of X , and a vector $R = (r_x)_{x \in X}$ (r_x is a non-negative integer for each $x \in X$) in the following way:

$X = V$, $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$, where for each i , $1 \leq i \leq n$, F_i is a multiset which contains two copies of each element in $N_G[v_i]$ and one copy of the set of elements which are at distance 2 from the vertex v_i in graph G , $r_x = 2$ for each $x \in X$.

Now we first prove the following correspondence.

Claim 5.2. *The set $D = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ is a disjunctive dominating set of G if and only if $\mathcal{C} = \{F_{i_1}, F_{i_2}, \dots, F_{i_k}\}$ is a 2-cover of X .*

Proof. (Proof of the claim) Suppose $D = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ is a disjunctive dominating set of G . Let $\mathcal{C} = \{F_{i_1}, F_{i_2}, \dots, F_{i_k}\}$. We want to show that \mathcal{C} is a 2-cover of X , that is, each element $v \in X$ is 2-covered by \mathcal{C} . Consider an arbitrary element $v \in X$. Note that $X = V$. If either v or one of its neighbor belongs to D , that is, $v_{i_r} \in N_G[v] \cap D$, then the set F_{i_r} contains 2 copies of v , and hence v is 2-covered. If $N_G[v] \cap D = \emptyset$, then $|N_G^2(v) \cap D| \geq 2$. Let $v_{i_p}, v_{i_q} \in N_G^2(v) \cap D$. Then each F_{i_p} and F_{i_q} contains a copy of v , and hence v is 2-covered. Hence \mathcal{C} is a 2-cover of X .

Conversely, suppose that $\mathcal{C} = \{F_{i_1}, F_{i_2}, \dots, F_{i_k}\}$ is a 2-cover of X . Let $D = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. We want to show that D is a disjunctive dominating set of G . Consider any arbitrary vertex $v \in V$. Then $v \in X$ (as $X = V$). Hence v is 2-covered by \mathcal{C} . Then, we have two possibilities: (i) There exists a set $F_{i_r} \in \mathcal{C}$, which contains 2 copies of v . In this case, v_{i_r} is either v or one of the neighbor of v , and hence v is disjunctively dominated by the set D . (ii) There exists two sets $F_{i_p}, F_{i_q} \in \mathcal{C}$, each containing a copy of v . Then v_{i_p} and v_{i_q} both are at distance 2 from the vertex v . Hence again v is disjunctively dominated by the set D . This proves that D is a disjunctive dominating set of G .

This completes the proof of the claim. \square

By the above claim, if D_d^* is a minimum cardinality disjunctive dominating set of G and \mathcal{C}^* is an optimal 2-cover of X , then $|D_d^*| = |\mathcal{C}^*|$. In [16], S. Rajgopalan and V. V. Vazirani gave a greedy

approximation algorithm for the CMSMC problem, which achieves an approximation ratio of $\ln(|F_M|) + 1$, where F_M is the maximum cardinality multiset in \mathcal{F} . Let \mathcal{C}^* be an optimal 2-cover and \mathcal{C}' be a 2-cover obtained by greedy approximation algorithm, then $|\mathcal{C}'| \leq (\ln(|F_M|) + 1) \cdot |\mathcal{C}^*|$. Given a 2-cover of X , we can also obtain a disjunctive dominating set of graph G of same cardinality. Suppose that D'_d is a disjunctive dominating set of G obtained from 2-cover \mathcal{C}' of X . Then $|D'_d| \leq (\ln(|F_M|) + 1) \cdot |D_d^*|$. If the maximum degree of the graph G is Δ , then the cardinality of a set in family \mathcal{C} will be at most $2(\Delta + 1) + \Delta(\Delta - 1)$, which is equal to $\Delta^2 + \Delta + 2$. Hence $|D'_d| \leq (\ln(\Delta^2 + \Delta + 2) + 1) \cdot |D_d^*|$. This completes the proof of the theorem. \square

5.2 Lower bound on approximation ratio

To obtain the lower bound, we give an approximation preserving reduction from the MINIMUM DOMINATION problem. The following approximation hardness result for the MINIMUM DOMINATION problem is already known.

Theorem 5.3. [5] *For a graph $G = (V, E)$, the MINIMUM DOMINATION problem can not be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$.*

Theorem 5.4. *For a graph $G = (V, E)$, MDDP can not be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(|V|^{O(\log \log |V|)})$.*

Proof. Let us describe the reduction from the MINIMUM DOMINATION problem to MDDP. Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ be an instance of the MINIMUM DOMINATION problem. Now, we construct a graph $H = (V_H, E_H)$ an instance of MDDP in the following way: $V_H = V \cup \{w_i, z_i \mid 1 \leq i \leq n\} \cup \{p, q\}$, $E_H = E \cup \{v_i w_i, w_i z_i, z_i p \mid 1 \leq i \leq n\} \cup \{pq\}$.

Fig. 1 illustrates the construction of the graph H from a given graph G . Note that $|V_H| = 3|V| + 2$.

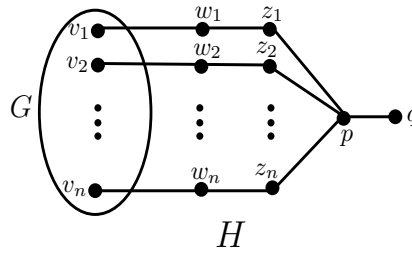


Figure 1: An illustration to the construction of H from G

If D^* is a minimum cardinality dominating set of G , then $D^* \cup \{p\}$ is a disjunctive dominating set of H . Hence for a minimum cardinality disjunctive dominating set D_d^* of H , $|D_d^*| \leq |D^*| + 1$.

On the other hand, let D_d be a disjunctive dominating set of H . Consider the vertex w_i . Since w_i is disjunctively dominated by the set D_d , one of the following possibilities may occur:

(i) $v_i \in D_d$, (ii) $w_i \in D_d$ or $z_i \in D_d$, (iii) $|N_H^2(w_i) \cap D_d| \geq 2$, that is, $p \in D_d$ and $N_G(v_i) \cap D_d \neq \emptyset$.

If (ii) occurs, then define $D_d = (D_d \setminus \{w_i, z_i\}) \cup \{v_i\}$. Do it for all i , $1 \leq i \leq n$. Note that the set $D = D_d \cap V$ dominates all the vertices of G , and $|D| \leq |D_d|$.

Now suppose that MDDP can be approximated with an approximation ratio of α , where $\alpha = (1 - \epsilon) \ln(|V_H|)$ for some fixed $\epsilon > 0$, by a polynomial time approximation algorithm APPROX-

DISJUNCTIVE. Let l be a fixed positive integer. Consider the following algorithm to compute a dominating set of a given graph G .

Algorithm 3: APPROX-DOMINATION(G)

Input: A graph $G = (V, E)$.

Output: A dominating set D of graph G .

begin

if *there exists a minimum dominating set D' of cardinality $\leq l$* **then**

$D = D'$;

else

 Construct the graph H ;

 Compute a disjunctive dominating set D_d of H using the algorithm

 APPROX-DISJUNCTIVE;

for $i = 1 : m$ **do**

if $w_i \in D_d$ **or** $z_i \in D_d$ **then**

$D_d = (D_d \setminus \{w_i, z_i\}) \cup \{v_i\}$;

$D = D_d \cap V$;

return D ;

Clearly, the algorithm APPROX-DOMINATION outputs a dominating set of G in polynomial time. If the cardinality of a minimum dominating set of G is at most l , then it can be computed in polynomial time. So, we consider the case, when the cardinality of a minimum dominating set of G is greater than l . Let D^* denotes a minimum cardinality dominating set of G , and D_d^* denotes a minimum cardinality disjunctive dominating set of H . Note that $|D^*| > l$.

Let D be the dominating set of G computed by the algorithm APPROX-DOMINATION, then $|D| \leq |D_d| \leq \alpha |D_d^*| \leq \alpha(|D^*| + 1) = \alpha(1 + \frac{1}{|D^*|})|D^*| < \alpha(1 + \frac{1}{l})|D^*|$.

Since ϵ is fixed, there exists a positive integer l such that $\frac{1}{l} < \epsilon$. So, $|D| < \alpha(1 + \epsilon)|D^*| = (1 - \epsilon)(1 + \epsilon) \ln(|V_H|)|D^*| = (1 - \epsilon') \ln(|V_H|)|D^*|$. Since $|V_H| = 3|V| + 1$, and $|V|$ is very large, $\ln(|V_H|) \approx \ln(|V|)$. Hence $|D| < (1 - \epsilon') \ln(|V|)|D^*|$. Hence, the dominating set D computed by the algorithm APPROX-DOMINATION achieves an approximation ratio of $(1 - \epsilon') \ln(|V|)$ for some $\epsilon' > 0$.

By Theorem 5.3, if the MINIMUM DOMINATION problem can be approximated within a ratio of $(1 - \epsilon') \ln(|V|)$, then $\text{NP} \subseteq \text{DTIME}(|V|^{O(\log \log |V|)})$. This proves that for a graph $H = (V_H, E_H)$, MDDP can not be approximated within a ratio of $(1 - \epsilon) \ln(|V_H|)$ unless $\text{NP} \subseteq \text{DTIME}(|V_H|^{O(\log \log |V_H|)})$. \square

5.3 APX-completeness

In this subsection, we prove that MDDP is APX-complete for bounded degree graphs. To prove this, we need the concept of L-reduction, which is defined as follows.

Definition 5.5. Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G , we say that f is an L-reduction if there are positive constants α and β such that for every instance x of F

1. $\text{opt}_G(f(x)) \leq \alpha \cdot \text{opt}_F(x)$.
2. for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can in polynomial time find a solution y' of x with $m_F(x, y') = c_1$ such that $|\text{opt}_F(x) - c_1| \leq \beta |\text{opt}_G(f(x)) - c_2|$.

To show the APX-completeness of a problem $\Pi \in \text{APX}$, it is enough to show that there is an L-reduction from some APX-complete problem to Π [3].

By Theorem 5.1, it is clear that MDDP can be approximated within a constant factor for bounded degree graphs. Thus the problem is in APX for bounded degree graphs. To show the APX-hardness of MDDP, we give an L-reduction from the MINIMUM VERTEX COVER PROBLEM (MVCP) for 3-regular graphs which is known to be APX-complete [1].

Theorem 5.6. *The MINIMUM DISJUNCTIVE DOMINATION PROBLEM is APX-complete for bipartite graphs with maximum degree 3.*

Proof. To show the APX-completeness of MDDP, it is enough to construct an L-reduction f from the instances of MVCP to the instances of MDDP. Given a graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$, and $E = \{e_1, e_2, \dots, e_m\}$, we construct a graph $H = (V_H, E_H)$ by replacing each edge $e_i = v_r v_s$ with the gadget H_i as shown in Figure 2. Clearly, H is a bipartite graph and maximum degree of H is 3.

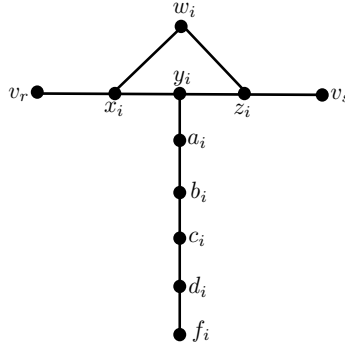


Figure 2: Graph H_i

Now, we first prove the following claim:

Claim 5.7. *Let D_d be a disjunctive dominating set of H of cardinality at most k . Then, there exists a disjunctive dominating set, say D'_d , of H of cardinality at most k such that $\{y_i, d_i \mid 1 \leq i \leq m\} \subseteq D'_d$ for each $i \in \{1, 2, \dots, m\}$. In addition, for each edge e_i in graph G , at least one of the end point of e_i is present in D'_d .*

Proof. (Proof of the claim) For some i , if $d_i \notin D_d$, then f_i must belong to D_d . In that case, if we remove f_i from the set D_d and add d_i in the set D_d , then D_d still remains a disjunctive dominating set of H .

So, we assume that $\{d_i \mid 1 \leq i \leq m\} \subseteq D_d$. Now, to disjunctively dominate the vertex b_i , at least one vertex from the set $\{c_i, b_i, a_i, y_i\}$ must belong to D_d . If $y_i \notin D_d$, then remove a vertex from the set $\{c_i, b_i, a_i\} \cap D_d$ from D_d and add y_i in D_d . Clearly, D_d still remains a disjunctive dominating set of H of same cardinality. Hence, given a disjunctive dominating set, say D_d , we can always construct a disjunctive dominating set, say D'_d , such that $\{d_i, y_i \mid 1 \leq i \leq m\} \subseteq D'_d$ and $|D'_d| \leq |D_d|$.

Now, we start with a disjunctive dominating set, say D_d , such that $\{d_i, y_i \mid 1 \leq i \leq m\} \subseteq D_d$. Let $S = \{y_i, d_i \mid 1 \leq i \leq m\}$, and v_r, v_s are end points of edge e_i in graph G . The set S disjunctively dominates all the vertices of H except the w'_i s. Also, for each w_i , S contains a vertex which is at distance two from w_i . Then, to disjunctively dominate the vertex w_i in graph H , at least one vertex from the set $\{w_i, x_i, z_i, v_r, v_s\}$ must belong to D_d . Now, if w_i, x_i or z_i belong to D_d , then remove them from D_d , and add either v_r or v_s in D_d . The resultant set D_d still remains a disjunctive dominating set of H of same or less cardinality. Note that, for each edge e_i , one of the endpoint is contained in the modified set D_d . This completes the proof of the claim. \square

Claim 5.8. *G has a vertex cover of cardinality at most k if and only if H has a disjunctive dominating set of cardinality at most $k + 2m$.*

Proof. (Proof of the claim) Let C be a vertex cover of G of cardinality at most k . Then, it can be easily verified that $D_d = C \cup \{y_i, d_i \mid 1 \leq i \leq m\}$ is a disjunctive dominating set of H of cardinality $k + 2m$.

Conversely, suppose that D_d is a disjunctive dominating set of H of cardinality at most $k + 2m$. Then by Claim 5.7, we may assume that $\{y_i, d_i \mid 1 \leq i \leq m\} \subseteq D_d$, and for each edge e_i in graph G , at least one of the end point of e_i is contained in the set D_d . Thus, $D_d \cap V$ is a vertex cover of G of cardinality at most k . \square

From Claim 5.7 and Claim 5.8, any disjunctive dominating set D_d of H can be transformed into a vertex cover C of G of cardinality at most $|D_d| - 2m$. Let D_d^* be a minimum disjunctive dominating set of H and C^* be a minimum vertex cover of G , then $|C^*| = |D_d^*| - 2m$. Hence, we have $||C| - |C^*|| \leq ||D_d| - |D_d^*||$. On the other hand, since G is a 3-regular graph, $m \leq 3|V_c^*|$. Hence $|D_d^*| = |V_c^*| + 2m \leq 7|V_c^*|$.

Hence f is an L-reduction with $\alpha = 7$ and $\beta = 1$. \square

6 Conclusion

In this article, we have proposed a linear time algorithm for MDDP in proper interval graphs. We have also tightened the NP-completeness of DDDP by showing that it remains NP-complete even in chordal graphs. From approximation point of view, we have proposed an approximation algorithm for MDDP in general graphs and have shown that this problem is APX-complete for bipartite graphs with maximum degree 3. Note that, the results presented in this paper, can easily be extended to b -disjunctive dominating set for $b \geq 3$. It would be interesting to study the complexity of this problem in other graph classes and also the relation between disjunctive domination number and other domination parameters.

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